<u>Last time</u> // overview - Gauss curvature of surfaces 2 things.

r distance function

$\mathcal{B} = \frac{1}{2} g^{-1} \mathcal{L}_{\partial_r} g$ shape operator of level sets

Fund eqn

$$\mathcal{L}_{\partial_r} \mathcal{B} + \mathcal{B}^2 = -R_{\partial_r}$$

where $R_{\partial_r}(X) = -R(\partial_r, X)\partial_r$

why 2 minuses? recall

$$-\int_{\Omega} K \, dvol = \int_{\partial\Omega} \langle D e_1, e_2 \rangle$$

$A^2,$

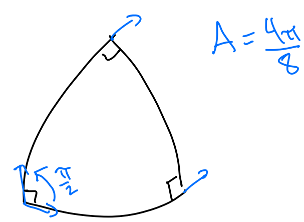
$\Omega^2, (e_1, e_2)$ is the only natural ordering

counterclockwise $\angle$ of ~~rotation~~ of an oriented frame, when traversed counterclockwise

$\uparrow$ Ω orientation

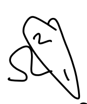
unfortunately, $-K$. To justify, K is d(rotation of <u>parallel</u> <u>relative</u> to frame)

on S^2



So if $\partial_r = e_1, X = e_2,$

$$\langle X, R_{\partial_r}(X) \rangle = -\langle R(e_1, e_2)e_1, e_2 \rangle = K \, dvol(e_1, e_2)$$



counterclockwise rotation of frame

Rmk: $R_{abcd} = g_{ce} R_{ab}{}^{e}{}_{d}$ is very symmetric

obvious: $R_{\underset{\leftrightarrow}{ab}cd}$.

& less obvious: $R_{ab\underset{\leftrightarrow}{cd}}$

⌐ tensoriality ⇒ suffices to check for on. frame $\langle e_c, e_d \rangle$

A skew sym ⇒
$dA + A \wedge A$ skew-sym.

Alt: compute $(\partial_i \partial_j - \partial_j \partial_i)\langle e_c, e_d \rangle$ ⌟

even less

Prop (Bianchi) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$ (⇐ tor-free)

⌐ here, it's important we don't use a frame

$\nabla_i \nabla_j \partial_k$ sym

6 permutations of (X,Y,Z)
only 3 distinct

$\nabla_\cdot \nabla_\cdot \cdot : \Sigma_3 \to \mathbb{R}$ in group algebra

$(1-(12))\nabla_\cdot \nabla_\cdot \cdot = R(\cdot,\cdot)\cdot$ $\quad \Sigma_3 \circlearrowright \mathbb{R}(\Sigma_3)$ regular rep'n

$R \in \mathrm{Fix}(23)$

$$(1 + (123) + (123)^2)(1 - (12)) = \sum_{g \in \Sigma_3} \mathrm{sgn}(g)\, g$$

$$= (1 + (123) + (123)^2)(1 - (23))$$ ⌟

Cor $R_{abcd} = R_{cdab}$

⌐ Algebra in Σ_4

$R(\sigma(X,Y,Z,W)) : \Sigma_4 \to \mathbb{R}$

$\sigma = (123)$
$\tau = (1234)$

$(1 + (123) + (123)^2)R = 0$
$(1 + (12))R = 0$
$(1 + (34))R = 0$

$(1 + \tau + \tau^2 + \tau^3)(1 + \sigma + \sigma^2)$

Representations of S^4

	dim
triv	1
alt	1
std	3
std⊗alt	3
$\boxed{\text{Fun}}$	$\boxed{2}$

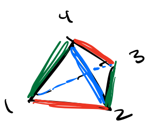
rule out • triv by $1+(12) = 2$

Bianchi → • alt by $(1+\sigma+\sigma^2) = 3$ doesn't triv

just rules this out

• std by $(12)(34) = 1\ 1\ 1$

• std ⊗ alt by $(12)(34) = 1\ 1\ 1$



$S_{(1234)} \longrightarrow S_{(rgb)} \subset \mathbb{R}^2$

Also useful

2nd Bianchi:

$$d^\nabla \Omega = 0$$

True in general, but there's a trick for R

$$d\left(dA + \tfrac{1}{2}[A,A]\right) + [A, dA + \tfrac{1}{2}[A,A]]$$

$$= \underbrace{\tfrac{1}{2}[dA,A] - \tfrac{1}{2}[A,dA] + [A,dA]}_{0} + \underbrace{\tfrac{1}{2}[A,[A,A]]}_{\text{Jacobi}}$$

Normal coordinate trick:

- Metric in normal coords

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \implies \Gamma_{\text{sym}} = 0$$

$$\implies \Gamma = 0$$

$$\implies g_{ij} = \delta_{ij} + O(|x|^2)$$

Use normal coords when you can to drop terms!

$$\nabla_a\nabla_b\nabla_c\partial_d - \nabla_b\nabla_a\nabla_c\partial_d = R(a,b)(\nabla_c\partial_d) = 0 \quad !$$

$$\implies (1+(123)+(123)^2)\nabla_a R_{bcde} = 0$$

Think: In $V = \mathbb{R}^{\{\text{sym}\{(a,b,c,d)\},\,(12;34)\}}$,

the ρ_2-isotypic part is 4 dim'l

$$\Sigma_4 \circlearrowright V \circlearrowleft \text{Sym}\{(a,b,c,d)\}$$

More fundamentally,

$$\begin{array}{c} \Sigma_4 \circlearrowright \otimes^4 TM \circlearrowleft GL(TM) \\ \big\downarrow R \\ \mathbb{R} \end{array}$$

ρ_2-part is an irred $GL(TM)$ rep'n but still many inequivalent lin. ~~reps~~ in it.

Alternative perspective

- $R : \Lambda^2 TM \longrightarrow \Lambda^2 TM$ self-adjoint s.t. R orthogonal to $\Lambda^4 TM$

(this is the irred GL-rep'n

$$\ell(\wedge) \perp R$$

Plucker: $\wedge$ exactly cuts out the simple tensors.

Sectional curvatures

For surfaces M^2

$T^{\text{Riem}} TM$ is 1-dimensional

$(R_{1212}$ determines all$)$

If $M \hookrightarrow \mathbb{R}^3$, it's $\det(h_{ab})$

Check

$$S = g^{ab} R_{ca}{}^{c}{}_{b}$$

$$= R_{12}{}^{1}{}_{2} + R_{21}{}^{2}{}_{1} = 2K$$

Return to general n:

If $\Pi \subseteq T_pM$ is a 2-plane, define

$$S_\pi = \exp_p(\Pi) \subseteq M$$

$$\sec(\Pi) = K_p(S_\pi).$$

$$\overset{\shortparallel}{\sec}(v,w)$$

Prop $\quad \sec(v,w) = \dfrac{v^a w^b v^c w^d R_{abcd}}{|v \wedge w|^2}$

$\ulcorner$ $\mathrm{II} = 0$ at p

$$\Longrightarrow R|_{TS_\pi} = \tilde{R}|_{TS_\pi} \text{ at } p$$

If (v,w) orthonormal, we are done

Else check effect of $\begin{pmatrix} av+bw \\ cv+dw \end{pmatrix}$. $\lrcorner$

i.e. quadratic form determined up to $\Lambda^4 V$ by restriction to simple tensors in $\Lambda^2 V$

Prop ~~The~~ All sectional curvatures determine R.

Pf. Let $D = R^{(1)} - R^{(2)}$.

$$0 = D(v+w, x, v+w, x)$$
$$= 2D(v, x, w, x)$$
$$0 = D(v, x+u, w, x+u)$$
$$= D(v, x, w, u) + D(v, u, w, x)$$

gives extra relation, makes it 0

Prop $Rc(v,v) = \sum_{e_i} R(e_i, v, e_i, v)$ for a on. basis $e_i \perp v$

$$= |v|^2 \sum_{e_i} \sec(e_i, \tfrac{v}{|v|})$$

$$S = \sum_{i \neq j} R(e_i, e_j, e_i, e_j)$$
$$= 2 \sum_{i<j} \sec(e_i, e_j)$$

Cor sec always ≥ 0 / ≤ 0 $\Rightarrow$ Ric, S always ≥ 0 / ≤ 0

In particular, constant sectional curvature $K \iff$

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$$

wait — is this Riemann?

Rmk $g_{ac}g_{bd}$ not Riemann, but maybe is when you skew-symmetrize.